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Nonlinear Chebyshev Approximation by H-Polynomials

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1. INTRODUCTION

In a previous note [4] we introduced "Horner-like" polynomials as approximating functions in Chebyshev approximation. The announced detailed presentation is the subject of this paper. The motive for considering those polynomials is mainly their applicability for the representation of functions in computer subroutines. We do not expect an elegant theory of approximation by these polynomials since they do not belong to the class of asymptotically convex functions [2] or to the class of varisolvent functions [3].

2. APPROXIMATION BY A CLASS OF RATIONAL FUNCTIONS

Let us assume that in a computer program the values of a function $f(x)$ are wanted, where $f(x)$ is real-valued, defined in a real interval $[\alpha, \beta]$,and is not a rational function. Then $f(x)$ cannot be computed by a finite number of elementary operations. Therefore, one has to find an approximation to $f(x)$ from the class of rational functions. It is, however, not necessary to restrict oneself to those rational functions which are represented as quotients of two polynomials whose coefficients are parameters of the approximation.

We first define a general class of approximating functions where the criterion of approximation involves data of the employed computer: Let A be the computing time for an addition or subtraction, M and D the computing time for a multiplication and division, respectively. We denote by $z_0 = x$ the argument and by $z_1 = c_1$, $z_2 = c_2$,..., and $z_q = c_q$ a collection of parameters. z_N is generated by a finite sequence of elementary operations

$$z_k = z_{i(k)} o_k z_{j(k)} \qquad (k = q + 1,..., N).$$

Here o_k means $+$, $-$, $\times$, or $\div$, and we restrict the integers i and j to $0 \leqslant i(k) \leqslant k-1$ and $0 \leqslant j(k) \leqslant k-1$. Then $z_N = p(x)$ is a rational function which requires the computing time

$$R = aA + mM + dD,$$

if a of the o_k are $+$ or $-$, m of them are $\times$, and d are $\div$. If the function $f(x)$, defined in the interval $[\alpha, \beta]$, and a real number $\bar{R} > 0$ are given, we have the approximation problem: Determine integers q, N, operations $o_{q+1}, \ldots, o_N$, functions $i(k)$ and $j(k)$ and real numbers $c_1, \ldots, c_q$ so that $R \leqslant \bar{R}$ and

$$\sup_{\alpha \leqslant x \leqslant \beta} |f(x) - p(x)|$$

is as small as possible. The dual problem would be: For a prescribed precision $\epsilon > 0$, determine q, N, o_k, $i(k)$, $j(k)$ and c_k, so that $|f(x) - p(x)| \leqslant \epsilon$ in $[\alpha, \beta]$ and R is as small as possible.

The appearance of computer data in the approximation criterion may be disadvantageous. A reasonable restriction would be to require $d = 1$, $m \leqslant \bar{m}$, $a \leqslant \bar{m}$, to prescribe $\bar{m}$ and to look for a best approximation $z_N = p(x)$. On the other hand, a possible generalization should be mentioned. We could include decision operations and allow alternatives depending upon the sign of certain intermediate results z_k (a generalization of segment approximation [1]).

3. H-POLYNOMIALS

In this paper we will not consider the general class of rational functions as defined in Section 2, but rather the subclass of the polynomials introduced in [4], which we call H-polynomials. In the definition given previously we now have $d = 0$, $a = m$, and o_k alternately $+$, $-$, and $\times$.

DEFINITION. An H-polynomial is a function $z_n(x)$ of the real variable x and the real parameters $a_0, a_1, \ldots, a_n$, generated by the following rules. Let $j = j(k)$ be a function with the properties:

$$j, k \text{ are integers}, \quad 2 \leqslant k \leqslant n, \quad 1 \leqslant j \leqslant k-1, \tag{1}$$

$$\left.\begin{aligned} &z_n(x) \text{ is recursively defined by } z_1(x) = a_0 x + a_1, \\ &z_k(x) = \begin{cases} z_{k-1}(x)\, x + a_k & \text{if } j(k) = 1 \\ \pm z_{k-1}(x)\, z_{j(k)} + a_k & \text{if } j(k) > 1 \end{cases} \quad (k = 2, \ldots, n) \end{aligned}\right\} \tag{2}$$

In the case $j(k) \equiv 1$, we get as $z_n(x)$ the polynomial

$$p_n(x) = a_0 x^n + \cdots + a_{n-1} x + a_n,$$

generated by the Horner-algorithm. Thus, the ordinary polynomials are contained in the class of H-polynomials. For $n = 3$ we get besides $p_3(x) = z_3^{(3)}(x)$, just one H-polynomial of fourth degree

$$z_3^{(4)}(x) = \pm(a_0x^2 + a_1x + a_2)^2 + a_3 .$$

For $n = 4$ we have 6 H-polynomials

$$\begin{aligned}
z_4^{(4)}(x) &= p_4(x), \\
z_4^{(5.1)}(x) &= z_3^{(4)}(x)\, x + a_4 , \\
z_4^{(5.2)}(x) &= \pm z_3^{(3)}(x)\, p_2(x) + a_4 , \\
z_4^{(6.1)}(x) &= \pm(z_3^{(3)}(x))^2 + a_4 , \\
z_4^{(6.2)}(x) &= \pm z_3^{(4)}(x)\, p_2(x) + a_4 , \\
z_4^{(8)}(x) &= \pm(z_3^{(4)}(x))^2 + a_4 .
\end{aligned}$$

The upper index denotes the degree of the polynomial, a second upper index numbers different polynomials of same degree. For $n \geqslant 2$ we have $(n - 1)!$ different H-polynomials, just as many as there are different functions $j(k)$. We can compare this with the approximation by ordinary rational functions. There we have in the case of n parameters n different types of approximating functions according to the degrees of the numerator and the denominator.
It is also possible to give a definition where $a_0x + a_1$ may be used as intermediate polynomial $z_{j(k)}(x)$. However, the foregoing definition has been chosen with a view to the following considerations.

4. Distinctness of the H-Polynomials

We can show that the H-polynomials generated by different functions $j(k)$ are essentially different. That means that there are no equivalences as in the case of the function

$$(a_0x^2 + a_1x + a_2)(a_0x^2 + a_1x) + a_3 ,$$

which is generated in a similar way as $z_3^{(4)}$ by using intermediate polynomials of the Horner-algorithm and can be written as

$$(a_0x^2 + a_1x + a_2/2)^2 - a_2^2/4 + a_3 ,$$

this being equivalent to $z_3^{(4)}(x)$.

By $g(k)$ $(1 \leqslant k \leqslant n)$ we denote the degree of $z_k(x)$. From (2) we find

$$g(1) = 1, \qquad g(k) = g(k-1) + g(j(k)) \quad (k = 2,..., n).$$

We define $b_{\nu k}$ as functions of the parameters $a_0, ..., a_k$ by writing

$$z_k(x) = \sum_{\nu=0}^{g(k)} b_{\nu k}(a_0, ..., a_k)\, x^\nu \qquad (k = 1,..., n),$$

and denote by F_{k-1} the class of functions which depend only upon $a_0, a_1, ..., a_{k-1}$.

By induction on k $(1 \leqslant k \leqslant n)$ we can show:

(i) $b_{g(k),k} = \sigma_k a_0^{\mu_k}$ with $\sigma_k = \pm 1$, $\mu_1 = 1$, $\mu_k = \mu_{k-1}$ for $j(k) = 1$, and $\mu_k = \mu_{k-1} + \mu_{j(k)}$ for $j(k) > 1$ $(k \geqslant 2)$.

(ii) $b_{\nu k} \in F_{k-1}$ $(1 \leqslant \nu \leqslant g(k))$,

(iii) $b_{ok} = a_k + c_{ok}$ with $c_{ok} \in F_{k-1}$.

By induction on m $(k \leqslant m \leqslant n)$ we can show:

(iv) $b_{\nu m} \in F_{k-1}$ $(g(m) - g(k) + 1 \leqslant \nu \leqslant g(m))$,

(v) $b_{g(m)-g(k),m} = \sigma_m \sigma_k p_{mk} a_0^{\mu_m - \mu_k} a_k + c_{km}$, with $p_{mk} > 0$ and integer, $c_{km} \in F_{k-1}$.

(iv) and (v) for $m = k$ are (ii) and (iii), where we have put $p_{kk} = 1$. The only nontrivial proof is that of (v).

We distinguish the following cases:

(a) $j(m) = 1$: $z_m(x) = z_{m-1}(x) \cdot x + a_m$, $g(m) = g(m-1) + 1$. We have $b_{g(m)-g(k),m} = b_{g(m-1)-g(k),m-1}$, thus, (v) with $p_{mk} = p_{m-1,k}$, since, according to (i) $\sigma_m = \sigma_{m-1}$, $\mu_m = \mu_{m-1}$.

(b) $j(m) > 1$: $z_m(x) = \pm z_{m-1}(x)\, z_{j(m)}(x)$, $g(m) = g(m-1) + g(j(m))$. According to (i), we have $\sigma_m = \pm \sigma_{m-1}\sigma_{j(m)}$, $\mu_m = \mu_{m-1} + \mu_{j(m)}$.

(b1) $1 < j = j(m) < k$. We have

$$\begin{aligned} b_{g(m)-g(k),m} &= \pm b_{g(m-1)-g(k),m-1} b_{g(j),j} + \cdots \\ &= \pm(\sigma_{m-1}\sigma_k p_{m-1,k} a_0^{\mu_{m-1}-\mu_k} a_k + \cdots)\, \sigma_j a_0^{\mu_j} + \cdots \\ &= \sigma_m \sigma_k p_{m-1,k} a_0^{\mu_m - \mu_k} a_k + \cdots, \end{aligned}$$

where $+ \cdots$ denotes functions from F_{k-1}. Thus, we have (v) with $p_{mk} = p_{m-1,k}$.

(b2) $k \leqslant j = j(m) \leqslant m - 1$. We have

$$\begin{aligned} b_{g(m)-g(k),m} &= \pm[b_{g(m-1)-g(k),m-1} b_{g(j),j} + b_{g(m-1),m-1} b_{g(j)-g(k),j}] + \cdots \\ &= \pm[(\sigma_{m-1}\sigma_k p_{m-1,k} a_0^{\mu_{m-1}-\mu_k} a_k + \cdots)\, \sigma_j a_0^{\mu_j} + \cdots \\ &\quad + \sigma_{m-1} a_0^{\mu_{m-1}} (\sigma_j \sigma_k p_{jk} a_0^{\mu_j - \mu_k} a_k + \cdots)] + \cdots \\ &= \sigma_m \sigma_k a_0^{\mu_m - \mu_k} (p_{m-1,k} + p_{jk})\, a_k + \cdots, \end{aligned}$$

thus, (v) is valid with $p_{mk} = p_{m-1,k} + p_{jk}$.

Now we assume that $j_1(k)$ and $j_2(k)$ are different functions with properties (1), $g_1(k)$ and $g_2(k)$ are the corresponding degree functions and $z_n^{(1)}$ and $z_n^{(2)}$ the H-polynomials generated by $j_1(k)$ and $j_2(k)$, respectively. There is a $\tilde{k}$ ($3 \leqslant \tilde{k} \leqslant n$) such that

$$j_1(k) = j_2(k) \qquad \text{for} \quad k < \tilde{k}, \qquad j_1(\tilde{k}) \neq j_2(\tilde{k}),$$

and we may assume that $j_1(\tilde{k}) < j_2(\tilde{k})$. Then we have $g_1(k) = g_2(k)$ for $k < \tilde{k}$ and $g_1(\tilde{k}) < g_2(\tilde{k})$. If $z_n^{(1)}$, $z_n^{(2)}$ have different degrees, we consider them as essentially different. Thus, we still have to treat the case $g_1(n) = g_2(n) = g$.

We write

$$z_n^{(i)}(x) = \sum_{\nu=0}^{g} b_{\nu n}^{(i)}(a_0, \ldots, a_n)\, x^\nu \qquad (i = 1, 2).$$

From (iv) it follows that $b_{\nu n}^{(1)}, b_{\nu n}^{(2)} \in F_{\tilde{k}-1}$ for $g - g_1(\tilde{k}) + 1 \leqslant \nu \leqslant g$ and also $b_{g-g_1(\tilde{k}),n}^{(2)} \in F_{\tilde{k}-1}$, but from (v), we have

$$b_{g-g_1(\tilde{k}),n}^{(1)} = \sigma_n \sigma_{\tilde{k}} p_{n\tilde{k}} a_0^{\mu_n - \mu_{\tilde{k}}} a_{\tilde{k}} + c_{\tilde{k}n},$$

with $c_{\tilde{k}n} \in F_{\tilde{k}-1}$.

We show that there is no differentiable transformation $a_k' = a_k'(a_0, \ldots, a_n)$ ($k = 0, \ldots, n$) of the parameters $a_0, \ldots, a_n$ such that the matrix

$$A = (\partial a_k' / \partial a_j)_{k,j=0,\ldots,n}$$

is nonsingular for $a_0 \neq 0$ and

$$b_{\nu n}^{(2)}(a_0', \ldots, a_n') = b_{\nu n}^{(1)}(a_0, \ldots, a_n)$$

is valid for $\nu = 0, 1, \ldots, g$. Such a transformation cannot exist, because for $a_0 \neq 0$ the rank of the matrix

$$B_1 = (\partial b_{\nu n}^{(1)} / \partial a_j)_{\nu=g,\ldots,g-g_1(\tilde{k});\, j=0,\ldots,n}$$

is $\tilde{k} + 1$, while the rank of the corresponding matrix

$$B_2 = (\partial b_{\nu n}^{(2)} / \partial a_j') = B_1 A$$

is $\leqslant \tilde{k}$. We formulate the results as a theorem.

THEOREM 1. *If $z_n^{(1)}(x)$, $z_n^{(2)}(x)$ are H-polynomials of the same degree with different generating functions $j_1(k), j_2(k)$, there is no differentiable transformation of the parameter space which is one-to-one (except possibly on the hyperplane $a_0 = 0$) and transforms $z_n^{(1)}$ into $z_n^{(2)}$.*

5. BEHAVIOR UNDER TRANSFORMATIONS

The first theorem answers the question whether an H-polynomial, multiplied by a constant, is an H-polynomial again. It is easy to see that, if $z_n(x)$ is an H-polynomial, then $-z_n(x)$ is also one. Therefore, it is sufficient to consider only constants greater than zero.

THEOREM 2. *Let $z_n(x, a_i)$ be an H-polynomial, $K > 0$ a constant. Then we have*

$$Kz_n(x, a_i) = z_n(x, \bar{a}_i) \quad (P)$$

with $\bar{a}_i = K^{(\mu_i/\mu_n)} \cdot a_i$, the μ_i defined as the exponent of a_0 in the first term of the intermediate polynomial

$$z_i(x, a_i) = \sigma_i a_0^{\mu_i} \cdot x^{g(i)} + \cdots, \qquad i = 1,..., n, \quad \textit{and} \quad \bar{a}_0 = K^{(\mu_1/\mu_n)} a_0 .$$

Proof. We prove $K^{(\mu_k/\mu_n)} \cdot z_k(x, a_i) = z_k(x, \bar{a}_i)$ (P_k) for $k = 1,..., n$ by induction on k. For $k = n$, $(P_n) = (P)$. For $k = 1$ we have

$$K^{(1/\mu_n)} \cdot z_1(x, a_i) = K^{(1/\mu_n)}(a_0 x + a_1) = \bar{a}_0 x + a_1 .$$

Thus, (P_1) holds.

Now assume (P_i) to be true for every $i < k$.

(a) $z_k(x, a_i) = z_{k-1}(x, a_i) \cdot x + a_k$. Here $\mu_{k-1} = \mu_k$. We get

$$\begin{aligned} K^{(\mu_k/\mu_n)} z_k(x, a_i) &= K^{(\mu_{k-1}/\mu_n)} \cdot z_{k-1}(x, a_i) \cdot x + K^{(\mu_k/\mu_n)} \cdot a_k \\ &= z_{k-1}(x, \bar{a}_i) \cdot x + \bar{a}_k . \end{aligned}$$

(b) $z_k(x, a_i) = \pm z_{k-1}(x, a_i) \cdot z_j(x, a_i) + a_k$. Here we have

$$\mu_k = \mu_{k-1} + \mu_j$$

and

$$\begin{aligned} K^{(\mu_k/\mu_n)} \cdot z_k(x, a_i) &= \pm K^{(\mu_{k-1}/\mu_n)} \cdot z_{k-1}(x, a_i) \cdot K^{(\mu_j/\mu_n)} \cdot z_j(x, a_i) + K^{(\mu_k/\mu_n)} \cdot a_k \\ &= \pm z_{k-1}(x, \bar{a}_i) \cdot z_j(x, \bar{a}_i) + \bar{a}_k . \end{aligned}$$

Thus, (P_k) holds in both cases, which completes the proof.

Now we will consider the behavior of H-polynomials under the transformation $x = Ay + B$; that means we ask the question if, $z_n(x)$ being an H-polynomial, $\bar{z}_n(y) = z_n(Ay + B)$ is also an H-polynomial. We shall see that this is not always true. However, we have the following theorem.

THEOREM 3. *Let $z_n(x, a_i)$ be an H-polynomial, $A > 0$ a constant. Then we have*

$$z_n(Ax, a_i) = z_n(x, \bar{a}_i) \quad (Q)$$

with $\bar{a}_i = A^{\tau_i} \cdot a_i$, $\tau_0 = (g(n)/\mu_n)$, $\tau_i = \mu_i \cdot \tau_0 - g(i)$. μ_i and $g(i)$ are defined as in Theorem 2.

Remark. It is sufficient to consider the case $A > 0$, since $\bar{z}_n(x) = z_n(-x, a_i)$ is evidently an H-polynomial of the same type.

Proof. We prove $z_k(x, \bar{a}_i) = A^{\tau_k} \cdot z_k(Ax, a_i)$ (Q_k) for $k = 1,..., n$ by induction on k. For $k = n$ is this property (Q). For $k = 1$ we find $\mu_1 = 1$, $g(1) = 1$, hence, $\tau_1 = \tau_0 - 1$.

$$z_1(x, \bar{a}_i) = A^{\tau_0} \cdot a_0 \cdot x + A^{\tau_1} \cdot a_1 = A^{\tau_1}(a_0 \cdot Ax + a_1) = A^{\tau_1} \cdot z_1(Ax, a_i).$$

Thus, (Q_1) holds.

Now assume (Q_i) to be true for every $i < k$.

(a) $z_k(x, \bar{a}_i) = z_{k-1}(x, \bar{a}_i) \cdot x + \bar{a}_k$. Here $\mu_k = \mu_{k-1}$, $g(k) = g(k-1) + 1$, $\tau_k = \tau_{k-1} - 1$.

$$\begin{aligned} z_k(x, \bar{a}_i) &= A^{\tau_{k-1}} \cdot z_{k-1}(Ax, a_i) \cdot x + A^{\tau_k} \cdot a_k \\ &= A^{\tau_{k-1}-1} \cdot z_{k-1}(Ax, a_i) \cdot Ax + A^{\tau_k} \cdot a_k \\ &= A^{\tau_k} \cdot z_k(Ax, a_i). \end{aligned}$$

(b) $z_k(x, \bar{a}_i) = \pm z_{k-1}(x, \bar{a}_i) \cdot z_j(x, \bar{a}_i) + \bar{a}_k$. Here $\mu_k = \mu_{k-1} + \mu_j$, $g(k) = g(k-1) + g(j)$, and, thus, $\tau_k = \tau_{k-1} + \tau_j$.

$$\begin{aligned} z_k(x, \bar{a}_i) &= \pm A^{\tau_{k-1}} \cdot z_{k-1}(Ax, a_i) \cdot A^{\tau_j} \cdot z_j(Ax, a_i) + A^{\tau_k} \cdot a_k \\ &= \pm A^{\tau_k} \cdot z_k(Ax, a_i). \end{aligned}$$

Thus, (Q_k) holds in both cases. This completes the proof.

Now we show, by giving an example, that $\bar{z}_k(y) = z_k(y + B)$ is not always an H-polynomial if $k > 3$. For $k = 4$ there are two H-polynomials of degree 5 (see Section 3):

$$z_4^{(5.1)}(x) = \pm[(a_0x^2 + a_1x + a_2)^2 + a_3] \cdot x + a_4$$

and

$$z_4^{(5.2)}(x) = \pm(a_0x^3 + a_1x^2 + a_2x + a_3)(a_0x^2 + a_1x + a_2) + a_4 .$$

The polynomial $P(x) = (x^3 + 2x + 3)(x^2 + 2) + 5$ is of the type $z_4^{(5.2)}$ with $a_0 = 1$, $a_1 = 0$, $a_2 = 2$, $a_3 = 3$, $a_4 = 5$. With $y = x - 1$ we find

$$\bar{P}(y) = P(y + 1) = y^5 + 5y^4 + 14y^3 + 25y^2 + 27y + 23.$$

It is easy to prove that there are no constants $a_0, ..., a_4$ such that $\bar{P}$ can be written in one of the forms $z_4^{(5.1)}$, $z_4^{(5.2)}$. Except for $z_4^{(5.1)}$ and $z_4^{(5.2)}$, all H-polynomials z_4 (see Section 3) are invariant under this transformation.

This shows a difference from the class P_n of common polynomials of degree $\leqslant n$. If $P(x)$ is such a polynomial then $K \cdot P(Ax + B) + C$, with any reals K, C, A, B is also in P_n. The property of being a best approximation to a given function is preserved under these linear transformations of the dependent and the independent variable. In general the last of these properties of invariance—translations in the variable x—is not possessed by H-polynomials. For practical use this is not necessarily a defect: $z_n(x + b)$ may in this case be used as approximating function with an additional parameter b. We remark that in case of rational functions (quotient of two polynomials) with the degree of the numerator less than the degree of the denominator, one has no invariance under the transformation $R(x) \to R(x) + C$.

6. Approximation by $z_3^{(4)}(x) = \pm (a_0x^2 + a_1x + a_2)^2 + a_3$

In this section we consider the approximation by the first nontrivial H-polynomial $z_3^{(4)}(x)$. This is a nonlinear approximation problem. As we have invariance under the transformations $x \to x + a$, we can choose the interval of approximation arbitrarily. If no special choice is made, we denote this interval by $[a, b]$. We set $\zeta = \{z_3^{(4)}(x) \mid (a_0, a_1, a_2, a_3) \in \mathbb{R}^4\}$.

(a) In the first instance we are interested in the question of whether there exists a best approximation in ζ for every function continuous on $[a, b]$. In [4] it was shown by an example that this is not always true.

It is, therefore, useful to extend the set ζ to a new set $\bar{\zeta}$ by adding certain functions in such a way that in $\bar{\zeta}$ a best approximation always exists. Explicitly, we will add to ζ the limits of all pointwise convergent sequences $\{f_\nu\} \subset \zeta$. In Theorem 5 it will be shown that the set $\bar{\zeta}$, which we get from ζ in this way, has property (Z) (for definition see part (b)) of degree 5. Hence, according to Rice [3, Lemmas 7–13], there exists a best approximation in $\bar{\zeta}$ for every function continuous on $[a, b]$.

The question is, which new functions does one get in this way. ζ is a subset of the linear space P_4, that is, the set of all polynomials of degree $\leqslant 4$. It is easy to see that P_4 is closed in the above sense. Therefore, we only get functions from P_4 as limits.

THEOREM 4. *Let P_2 be the set of all polynomials of degree $\leqslant 2$. Then* $\bar{\zeta} = \zeta \cup P_2$.

Proof. Let $P = b_0x^4 + b_1x^3 + b_2x^2 + b_3x + b_4$ be a polynomial in P_4, $z_{3,i} = (a_{0,i}x^2 + a_{1,i}x + a_{2,i})^2 + a_{3,i}$ a sequence of H-polynomials with $\| z_{3,i} - P \| \to 0$. We restrict ourselves to the case of a $+$ sign in $z_{3,i}$. The proof for the other case is similar.

From $\| z_{3,i} - P \| \to 0$ we get the following properties:

(i) $\lim a_{0,i}^2 = b_0$;

(ii) $\lim 2a_{0,i} \cdot a_{1,i} = b_1$;

(iii) $\lim\{a_{1,i}^2 + 2a_{0,i} \cdot a_{2,i}\} = b_2$;

(iv) $\lim 2a_{1,i} \cdot a_{2,i} = b_3$;

(v) $\lim\{a_{2,i}^2 + a_{3,i}\} = b_4$.

We distinguish two cases:

Case 1. The sequences $a_{k,i}$ are bounded for $k = 0,..., 3$. We show that this, with the exception of a trivial case, implies that the sequences $a_{k,i}$ converge to constants a_k and, hence, $z_{3,i}$ to an H-polynomial $z_3^{(4)}$. First, we exclude the trivial case that $b_0 = b_1 = b_2 = b_3 = 0$. In this case $z_{3,i}$ converges to a constant and, thus, to a $z_3^{(4)}$. From (i), results $\lim a_{0,i} = a_0 = (b_0)^{1/2}$. If $a_0 = 0$, we get $\lim a_{1,i} = a_1 = (b_2)^{1/2}$ from (iii), otherwise $\lim a_{1,i} = a_1 = (b_1/2a_0)$ from (ii). Hence, $a_{1,i}$ converges. $a_0 = a_1 = 0$ is possible only if $b_0 = b_1 = b_2 = b_3 = 0$. This was excluded. If $a_0 \neq 0$, it follows from (iii) that $\lim a_{2,i} = a_2 = (b_2 - a_1{}^2)/2a_0$. If $a_0 = 0$, hence, $a_1 \neq 0$, we have $\lim a_{2,i} = a_2 = (b_3/2a_{1,i})$, from (iv). Thus, $a_{2,i}$ converges, too. Finally, we get from (v) $\lim a_{3,i} = b_4 - a_2{}^2$. Hence, we have proved that, in Case 1, we always get functions in ζ.

Case 2. Not all sequences $a_{k,i}$ are bounded.

(i) implies that $a_{0,i}$ is always bounded. Assume $a_{1,i} \to \pm\infty$. Then it follows from (ii) that $a_{0,i} \to 0$ and from (iv) $a_{2,i} \to 0$. This implies $a_{1,i}^2 + 2a_{0,i} \cdot a_{2,i} \to \infty$ which contradicts (iii). Hence, $a_{1,i}$ must also be bounded. There remains the case $a_{2,i} \to \pm\infty$. From (iv) we get $a_{1,i} \to 0$. As $a_{0,i}$ is bounded, it follows from (ii) $b_1 = 0$. (iii) implies that $a_{1,i}^2 + 2a_{0,i} \cdot a_{2,i}$ is bounded. As $a_{1,i} \to 0$, this is only possible if $a_{0,i} \to 0$; hence, $b_0 = 0$. This shows that all limits are in P_2. On the other hand, let $P = b_2x^2 + b_3x + b_4$ be any polynomial in P_2. Set $a_{0,i} = (b_2/2i)$, $a_{1,i} = (b_3/2i)$, $a_{2,i} = i$, $a_{3,i} = b_4 - i^2$.

Then

$$a_{0,i}^2 = (b_2{}^2/4i^2) \to 0,$$
$$2a_{0,i} \cdot a_{1,i} = 2 \cdot (b_2 \cdot b_3/4i^2) \to 0,$$
$$a_{1,i}^2 + 2a_{0,i} \cdot a_{2,i} = (b_3{}^2/4i^2) + 2 \cdot (b_2/2i) \cdot i \to b_2\,,$$
$$2a_{1,i} \cdot a_{2,i} = 2 \cdot (b_3/2i) \cdot i = b_3\,,$$

and

$$a_{2,i}^2 + a_{3,i} = i^2 + b_4 - i^2 = b_4\,.$$

Thus, $z_{3,i} \to P$, which completes the proof.

(b) $\bar{\zeta}$ is not varisolvent. Regard the following common approximation problem: Let F be any subset of $C[a, b]$, $g \in C[a, b]$, $g \notin F$. Find a $f^* \in F$ such that $\| f^* - g \| = \min_{f \in F} \| f - g \|$.

We follow Rice [3] with the following definitions.

DEFINITION. F has property (Z) of degree m in $f \in F$, if for $g \neq f$, $g \in F$, $f - g$ has at most $m - 1$ zeros in $[a, b]$.

DEFINITION. F is locally solvent of degree m at $f \in F$ if given a set $X_m = \{x_i \mid i = 1,..., m; a \leqslant x_1 < x_2 < \cdots < x_m \leqslant b\}$ and $\epsilon > 0$, then there is a $\sigma(f, \epsilon, X_m) > 0$ such that $| Y_i - f(x_i)| < \sigma$, $i = 1,..., m$, implies that there exists a $g \in F$ with

$$g(x_i) = Y_i \qquad i = 1,..., m$$

and

$$\| f - g \| < \epsilon.$$

DEFINITION. If F is locally solvent and possesses property (Z) for each $f \in F$ with the same degree, then F is varisolvent.

THEOREM 5. *$\bar{\zeta}$ has property (Z) of degree 5 for each $z \in \bar{\zeta}$.*

Proof. We choose $[a, b] = [-1, 1]$. We have to show that for every $z^* \in \bar{\zeta}$ we can find a $z \in \bar{\zeta}$ such that $z^* - z$ has exactly four zeros on $[-1, 1]$. $z^* - z$ is a function in P_4. Hence, for $z \neq z^*$ there are not more than four zeros on $[-1, 1]$. We set $z(\alpha, \beta; x) = (\alpha(x^2 - \frac{1}{2}))^2 + \beta$. $z(\alpha, \beta; x) \in \bar{\zeta}$ for any reals α, β. Then

$$z(\alpha, \beta; -1) = z(\alpha, \beta; 0) = z(\alpha, \beta; 1) = \beta + (\alpha^2/4),$$

and

$$z(\alpha, \beta; -\tfrac{1}{2}(2)^{1/2}) = z(\alpha, \beta; \tfrac{1}{2}(2)^{1/2}) = \beta.$$

Take $\beta < \min_{x\in[-1,1]} z^*(x)$ and α such that $\beta + (\alpha^2/4) > \max_{[-1,1]} z^*(x)$. Then $z^*(x) - z(\alpha, \beta; x)$ has exactly four zeros on $[-1, 1]$.

THEOREM 6. *$\bar{\zeta}$ is not varisolvent.*

Proof. Let $z(x)$ be the polynomial of degree four with zeros at $-1, -\frac{1}{2}, \frac{1}{2}$, 1 and $z(0) = 1$. Then $z \in \bar{\zeta}$. By $z_d(x)$ we denote the polynomial of degree four which is equal to $z(x)$ for $x = -1, -\frac{1}{2}, 0, \frac{1}{2}$ and with $z_d(1) = d$. The coefficients of z_d are continuous in d, and, hence, the fourth zero of z_d is continuous in d. This implies the existence of a $d' > 0$ such that for $0 < | d | < d'$ the graph of $z_d(x)$ is not symmetrical about a line parallel to the y-axis (since the zeros are not), and, therefore, $z_d \notin \bar{\zeta}$ since all polynomials of degree four of $\bar{\zeta}$ have this symmetry. From this it follows that $\bar{\zeta}$ is not locally solvent of degree five in z, and, hence, $\bar{\zeta}$ is not varisolvent.

(c) $\bar{\zeta}$ is not asymptotically convex. First, we give an example of a function $f(x)$ for which the error function for the best approximation alternates only three times.

We approximate $f(x) = x^3$ ($\notin \bar{\zeta}$) in $[-1, 1]$. In P_2, $z^*(x) = \frac{3}{4}x$ is the best approximation with extremal points of the error curve at $x = -1, -\frac{1}{2}, \frac{1}{2}, 1$. We have $\| f - z^* \| = \frac{1}{4}$. We denote by G the following subset of the $x - y$-plane: $-1 \leqslant x \leqslant 1$, $x^3 - \frac{1}{4} \leqslant y(x) \leqslant x^3 + \frac{1}{4}$ (Fig. 2). We show that there are no functions in $\bar{\zeta}$, other than z^*, whose graphs are subsets of G. As G consists only of polynomials of degree exactly four or less than three, we have to consider only the first: $z = (a_0x^2 + a_1x + a_2)^2 + a_3$. Due to symmetry, it is sufficient to consider only the case of $+$ sign in z. According as $a_0x^2 + a_1x + a_2$ has no zeros or two, we have two types z (Fig. 1.)

It is easy to see that there is no function, other than z^*, with continuous second derivative and no turning-point in $[-1, 1]$, whose graph lies entirely in G. Thus, functions of type A can be disregarded. Considering the fact that the two minimum values in B are equal, we conclude that there are no functions of this type either with their graphs in G. This shows that $z^* = \frac{3}{4}x$ is really the best approximation in $\bar{\zeta}$.

Now consider the function $z_1(x) = -3[x^2 - \frac{1}{2}x - \frac{1}{2}]^2 + 1 \in \bar{\zeta}$. It is easy

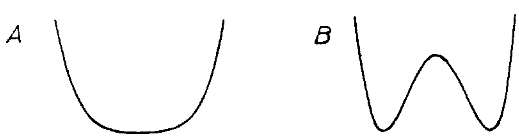

FIGURE 1

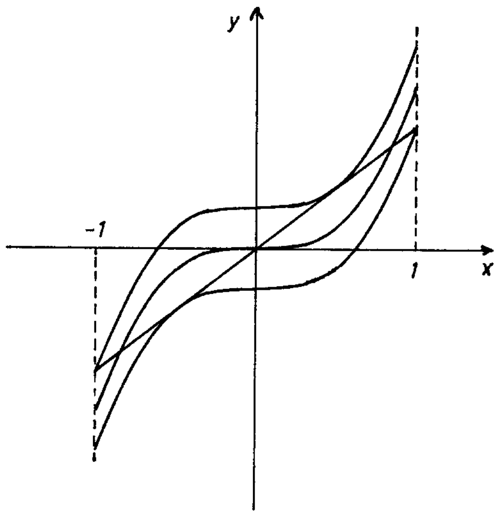

FIGURE 2

to see, that $z_1 - z^*$ has the same sign as $x^3 - z^*$ at the points $-1, -\frac{1}{2}, \frac{1}{2}, 1$. This means that the criterion of Kolmogoroff does not hold.

On the other hand, if $\bar{\zeta}$ is asymptotically convex this criterion must hold (see Meinardus [2, Satz 83]). Thus, $\bar{\zeta}$ is not asymptotically convex.

7. A SUFFICIENT CONDITION FOR RELATIVELY BEST APPROXIMATIONS

We state a specialized version of a sufficient condition for relative minima of the error

$$\rho(a) = \max_{[\alpha,\beta]} |f(x) - F(a, x)|,$$

which has been derived in [5]. The function $f(x)$ which is to be approximated by $F(a, x) = F(a_0, a_1, ..., a_n, x)$ in the interval $[\alpha, \beta]$ of the reals is assumed to possess a continuous second derivative in $[\alpha, \beta]$. $F(a, x)$ should also have continuous second derivatives in $x, a_0, ..., a_n$ for $\alpha \leqslant x \leqslant \beta$ and all real $a_0, ..., a_n$.

THEOREM. *Let* $\tilde{a} = (\tilde{a}_0, \tilde{a}_1, ..., \tilde{a}_n)$ *be given with*

$$\max_{[\alpha,\beta]} |f(x) - F(\tilde{a}, x)| = \epsilon,$$

where

$$f(x) - F(\tilde{a}, x) = \begin{cases} \epsilon & \textit{for} \quad x = x_1, ..., x_r \\ -\epsilon & \textit{for} \quad x = x_{r+1}, ..., x_p \end{cases}$$

(and possibly $|f - F| = \epsilon$ *at other points also). Assume that conditions* (1) *and* (2) *are fulfilled:*

(1) *There exist real numbers* $u_j > 0$ $(j = 1,..., r)$ *and* $u_j < 0$ $(j = r + 1,..., p)$ *with*

$$\sum_{j=1}^{p} u_j \frac{\partial F(\tilde{a}, x_j)}{\partial a_k} = 0 \qquad (k = 0,..., n).$$

(2) *For maxima and minima of the error function* $f(x) - F(\tilde{a}, x)$ *at interior points of the interval* $[\alpha, \beta]$, *i.e., for* x_j *with* $\alpha < x_j < \beta$, *we have*

$$\frac{d^2 f(x_j)}{dx^2} - \frac{\partial^2 F(\tilde{a}, x_j)}{\partial x^2} \neq 0.$$

(3) *If* $\sum_j'$ *denotes the sum over all j with* $\alpha < x_j < \beta$, *the quadratic form,*

$$\sum_{j=1}^{p} u_j \sum_{k,m=0}^{n} \frac{\partial^2 F(\tilde{a}, x_j)}{\partial a_k\, \partial a_m} \zeta_k \zeta_m + \sum_{j=1}^{p}{}' u_j \left(\frac{d^2 f(x_j)}{dx^2} - \frac{\partial^2 F(\tilde{a}, x_j)}{\partial x^2}\right)^{-1} \left(\sum_{k=0}^{m} \frac{\partial^2 F(\tilde{a}, x_j)}{\partial a_k\, \partial x} \zeta_k\right)^2,$$

is negative definite on the linear subspace of R^{n+1} *defined by*

$$\sum_{k=0}^{n} \frac{\partial F(\tilde{a}, x_j)}{\partial a_k} \zeta_k = 0 \qquad (j = 1,..., p).$$

Then we have

$$\rho(a) > \rho(\tilde{a}),$$

for all a in a neighborhood (e.g. spherical Euclidean) of $\tilde{a}$ *in* R^{n+1}.

Remark. The differentiability conditions could have been reduced in this special case.

8. Example

For the sake of the approximation by H-polynomials $z_3^{(4)}$ in [0, 1] a FORTRAN program has been written which determines local best approximations with five extremal points for the error curve; one of these at $x = 0$ and one at $x = 1$. By interpolation in the four points $x = 0.04, 0.3, 0.7, 0.96$ first estimations for local minima are computed which are improved by Newton's method. With the method of Section 7 it is then tested if one really has a local minimum.

The interpolation problem leads to an algebraic equation of the third degree; hence, there are not more than three solutions. For practical use we are above all interested in approximations with a deviation less than that we get by approximation with common polynomials of degree three. This depends naturally on the function which is to be approximated.

EXAMPLE. The approximation of $\cos[(\pi/4)(x)^{1/2}]$ in [0, 1]. If we approximate by polynomials of degree three we have a maximal deviation of 2.76×10^{-8}. Using the program we get two local minima with maximal errors 2.01×10^{-8} and 0.99×10^{-8}. The latter is given by

$$z = +(a_0x^2 + a_1x + a_2)^2 + a_3$$

with

$$\begin{aligned} a_0 &= 0.001\ 506\ 270\ 633\ 6432 \\ a_1 &= -0.107\ 356\ 277\ 464\ 62 \\ a_2 &= 1.436\ 454\ 524\ 983\ 8 \\ a_3 &= -1.063\ 401\ 612\ 612\ 8 \end{aligned}$$

This is a polynomial of type B (Fig. 1) with the three extremal points (35.6 − 17.8, −1.06), (35.6, −0.84) and (35.6 + 17.8, −1.06). The other local minimum is of type A with the minimum (13.6, −4.3).

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